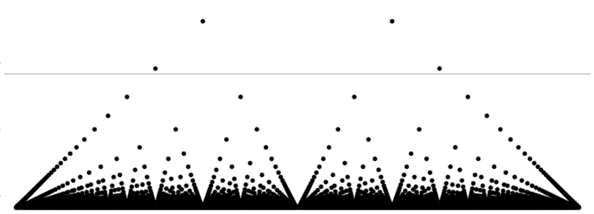
Math 451: Introduction to General Topology
Lecture 11

(6) Thomae's function. Let $T:(0,1)\to[0,\frac{1}{2}]$ be defined by

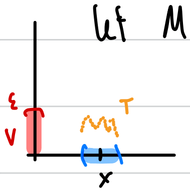
$$T(x):=\begin{cases}0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\\ \frac{1}{m} & \text{if } x\in\mathbb{Q} \text{ and } x=\frac{n}{m} \text{ is reduced.}\end{cases}$$



Claim. T is continuous at irrationals and discontinuous at rationals.

Proof. If $x=\frac{n}{m}$ is a rational and $\frac{n}{m}$ is reduced, then $T(x)=\frac{1}{m}$ and $V:=(0,\frac{2}{m})\ni\frac{1}{m}$ is open while $T^{-1}(V)$ only contains rationals, so it cannot contain an open interval $I\ni x$ because every nonempty open interval contains an irrational (HW).

Now let $x\in\mathbb{R}\setminus\mathbb{Q}$, so $T(0)=0$. Let $V:=(-\varepsilon,\varepsilon)$ be an open ball centered at 0.



Let $M\in\mathbb{N}^+$ be such that $\frac{1}{M}<\varepsilon$. But the set $F:=\{\frac{n}{m}: n<m \text{ and } m\leq M\}$ has at most M^2 elements, hence is finite, thus $\exists\,\delta>0$ s.t. $(x-\delta,x+\delta)\cap F=\emptyset$, so $T((x-\delta,x+\delta))\subseteq[0,\frac{1}{M})\subseteq V$. Thus T is continuous at x. □

Remark. In homework, we show that the set of continuity points of a function between metric spaces is G_δ. To see this on Thomae's function, note that $\mathbb{Q}$ is F_σ since $\mathbb{Q}=\bigcup_{q\in\mathbb{Q}}\{q\}$ and each $\{q\}$ is closed and $\mathbb{Q}$ is ctbl. Thus, $\mathbb{R}\setminus\mathbb{Q}$ is G_δ, hence

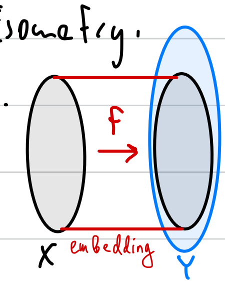
$$\mathrm{Cont}(T)=(\mathbb{R}\setminus\mathbb{Q})\cap(0,1) \text{ is } G_\delta.$$

Question. Does an opposite function exist? I.e. is there a function $f:(0,1)\to\mathbb{R}$ s.t. $\mathrm{Cont}(f)=\mathbb{Q}$?

Answer. No! The set $\mathrm{Cont}(f)$ is G_δ and we will show later that $\mathbb{Q}$ is not G_δ.

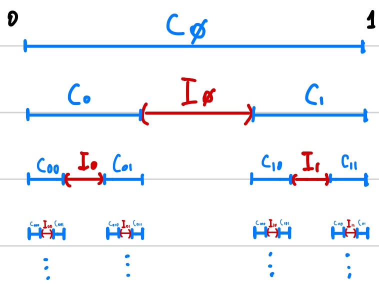
Homeomorphisms and embeddings.

Let X,Y be metric spaces. Call a function $f:X\to Y$ a homeomorphism if it is a bijection and both f and f^{-1} are continuous; in other words, f maps open sets to open sets and back (i.e. f^{-1} maps open sets to open sets). Note that an isometric bijection is a homeomorphism, but a homeomorphism need not be an isometry. Call $f: X \to Y$ an embedding if f is a homeomorphism from X to $f(X)$.



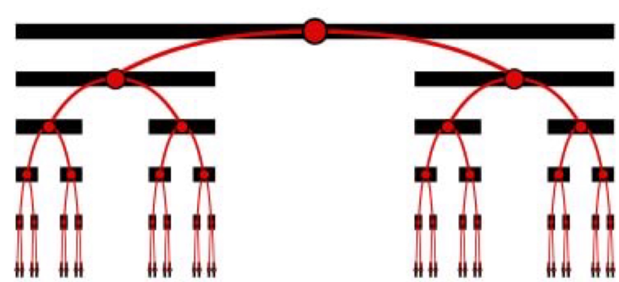
To give a nice example of a homeomorphism, we first define the standard Cantor set.

The standard Cantor set. The standard Cantor set is a closed subset $C \subseteq [0,1]$ defined as follows. We define a sequence $(C_s)_{s \in 2^{<\mathbb{N}}}$ of closed intervals and a sequence $(I_s)_{s \in 2^{<\mathbb{N}}}$ of open int. by induction as follows: let $C_\emptyset := [0,1]$. Then let $I_\emptyset :=$ the middle third open interval in the interval $C_\emptyset$. Then $C_\emptyset \setminus I_\emptyset = C_0 \sqcup C_1$ is a disjoint union of two closed intervals. Then I_0 and I_1 are the middle third open intervals of C_0 and C_1, respectively. In general, for each $s \in 2^{<\mathbb{N}}$, if C_s is already defined, then I_s is the middle third open interval in C_s and $C_s \setminus I_s = C_{s0} \sqcup C_{s1}$, where C_{s0} and C_{s1} are the two disjoint closed intervals, C_{s0} to the left of C_{s1}, whose union is $C_s \setminus I_s$.



Now the Cantor set C is defined as $C := \bigcap_{n \in \mathbb{N}} \bigsqcup_{s \in 2^n} C_s$. Then C is closed since $\forall n \in \mathbb{N}$, $\bigsqcup_{s \in 2^n} C_s$ is a finite union of closed sets, hence closed, so $\bigcap_{n \in \mathbb{N}} \bigsqcup_{s \in 2^n} C_s$ is an intersection of closed sets.

It is easy to see that all endpoints of the intervals C_s, $s \in 2^{<\mathbb{N}}$, are in C, so C is infinite. But is C ctbl? Are there other points?



Theorem. $2^{\mathbb{N}}$ and the Cantor set are homeomorphic. A homeomorphism is given by $f: 2^{\mathbb{N}} \to C$, where $f(x) :=$ the unique element of $\bigcap_{n \in \mathbb{N}} C_{(x|n)}$.

Proof. Firstly, $f(x)$ is well-defined by the completeness of $[0,1]$ because $C_{(x|_n)}$ is a decreasing sequence of closed sets of diameter $\operatorname{diam}(C_{(x|_n)}) = \frac{1}{3^n} \to 0$, hence $\bigcap_{n\in\mathbb{N}} C_{(x|_n)} \neq \emptyset$, thus it has a unique element since $\operatorname{diam}(\bigcap_n C_{(x|_n)}) = 0$.

Injectivity. If $x \neq y$ in $2^{\mathbb{N}}$, then $\exists$ least $n\in\mathbb{N}$ $x(n) \neq y(n)$, but then $f(x), f(y) \in C_s$ where $s := x|_n = y|_n$, but $f(x) \in C_{s x(n)}$ and $f(y) \in C_{s y(n)}$ and $C_{s x(n)}$ and $C_{s y(n)}$ are disjoint, so $f(x) \neq f(y)$.

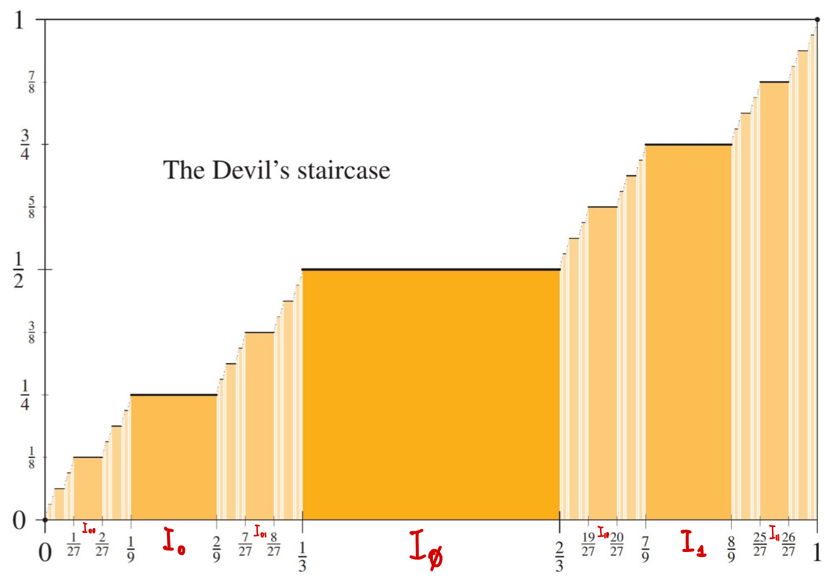
Surjectivity. If $r \in C$, then $r \in \bigcap_{n\in\mathbb{N}} \bigsqcup_{s\in 2^n} C_s$, so for each $n\in\mathbb{N}$, $r \in \bigsqcup_{s\in 2^n} C_s$. But this union is disjoint, so there is a unique $s_n \in 2^n$ with $r \in C_{s_n}$. But $C_{s_n} \supseteq C_{s_{n+1}}$ hence $s_n \subsetneq s_{n+1}$ so the "union" of the s_n gives an $x \in 2^{\mathbb{N}}$ with $x|_n = s_n$ $\forall n\in\mathbb{N}$. Thus, $f(x) =$ the unique element in $\bigcap_{n\in\mathbb{N}} C_{s_n} = \{r\}$, hence $f(x) = r$.

Continuity of f. We show that f is continuous at every $x \in 2^{\mathbb{N}}$. Fix $x \in 2^{\mathbb{N}}$, so $f(x) \in C$. Fix $V := B_\varepsilon(f(x))$. Then choosing $n\in\mathbb{N}$ large enough so that $\frac{1}{3^n} < \varepsilon$, we get that $f(x) \in C_{x|_n} \subseteq V$ because $\operatorname{diam}(C_{x|_n}) = \frac{1}{3^n}$. Denote $s := x|_n$, then $f([s]) \subseteq C_s \subseteq V$, so f is continuous at x.

Continuity of f^{-1}. HW. ☐

Remark. The homeomorphism f in this theorem is not an isometry: indeed, $x := 000\ldots$ and $y := 111\ldots$ have distance 2^{-1} in $2^{\mathbb{N}}$ but $f(x) = 0$ and $f(y) = 1$ have distance 1 in $\mathbb{R}$.

Another cool example of a continuous function.



Let $(I_s)_{s\in 2^{<\mathbb{N}}}$ be the sequence of open (middle third) intervals as in the definition of the Cantor set. Then $U := \bigcup_{s\in 2^{<\mathbb{N}}} I_s$ is open and $C = [0,1] \setminus U$. Define a function $f : U \to [0,1]$ by setting for each $s \in 2^{<\mathbb{N}}$, $f|_{I_s} := 0.s_0 s_1 \ldots s_{n-1} 1$, where the latter is the binary representation of a real. f is uniformly continuous on U hence can be uniquely extended to $\overline{U} = [0,1]$. Details in HW.

Space of bdd continuous functions

For metric spaces X, Y, recall that $B(X,Y)$ denotes the space of bdd functions, which is a metric space when equipped with the uniform metric d_u. Also recall that if Y is complete then $B(X,Y)$ is complete.

Let $BC(X,Y) := \{f \in B(X,Y) : f \text{ is continuous}\}$.

Prop. $BC(X,Y)$ is a closed subset of $B(X,Y)$, in the uniform metric d_u.
Equivalently, the uniform limit of bdd continuous functions is continuous.

Proof. Let $(f_n) \subseteq BC(X,Y)$ and suppose $f_n \to_{d_u} f$ for some $f \in B(X,Y)$. We need to show that $f \in CB(X,Y)$, i.e. f is continuous. Fix $x_0 \in X$ and show that f is continuous at x_0. To this end fix $\varepsilon > 0$ and we need to find $\delta > 0$ s.t. $x \in B_\delta(x_0)$, $f(x) \in B_\varepsilon(f(x_0))$. For $\varepsilon/3$, $\exists\, n \in \mathbb{N}$ s.t. $d_u(f, f_n) < \varepsilon/3$. $(*)$ Since f_n is continuous, there is $\delta > 0$ s.t. for all $x \in B_\delta(x_0)$, $(\dagger)$ $d(f_n(x), f_n(x_0)) < \varepsilon/3$. Then $\forall x \in B_\delta(x_0)$ we have:

$$d(f(x), f(x_0)) \leq \underbrace{d(f(x), f_n(x))}_{(*)} + \underbrace{d(f_n(x), f_n(x_0))}_{(\dagger)} + \underbrace{d(f_n(x_0), f(x_0))}_{(*)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

QED